B3 - Stochastic Processes : Back-Paper Exam

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June 6, 2025. Time : 10.00 AM - 12.30 PM. **Points : 50**

2 points will be deducted if you do not write your name on the answerscript.

ALL QUESTIONS CARRY 10 POINTS. ATTEMPT ALL FIVE OF THEM.

You are free to use any results that you have learnt in your probability courses but please cite them clearly. Provide as many details as you can. Some simple notions and notations are recalled at the end.

1. Let G_n be the graph with vertex set $V_n = \{-n, \ldots, n\}^d \subset \mathbb{Z}^d, d \geq 1$ and edges between $x, y \in V_n$ if $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i| = 1$. Let $p \in (0, 1)$. Define $G_n(p)$ to be the random subgraph of G_n with vertex set V_n and edges in G_n are retained independently with probability p (and deleted with probability $1 - p$). Let $I(n, p)$ denote the number of vertices of degree 2 in $G_n(p)$. Show that as $n \to \infty$,
$$(2n)^{-d} I(n, p) \to d(2d - 1)p^2(1 - p)^{2d-2}, \quad \text{a.s.}$$
2. Let $G_n = G(n, p)$ be the Erdős-Rényi random graph on n vertices with edge-probability p. Let $Cl(G_n)$ denote the size (in terms of vertices) of the largest clique in G_n. Show that
$$\lim_{n\to\infty} \mathbb{P}\{Cl(G_n) \geq 4\} = \begin{cases} 0 & \text{if } n^{2/3}p \to 0. \\ 1 & \text{if } n^{2/3}p \to \infty. \end{cases}$$
3. Consider Polya's urn starting with 1 red ball and 1 green ball. Let G_t be the number of green balls at time t. Let $M_t = \frac{G_t}{t+2}$. Show that M_t is a Martingale and it converges in distribution to a uniform random variable in $[0, 1]$. Does M_t converge a.s. ?
4. Let $W \subset V$ be a finite set and $\tau = \tau_{W^c}$ be the hitting time of W^c for X_t, an irreducible HMC. Show that $\tau < \infty$ a.s. and further if h is a harmonic function on W, then show that $h(X_{t\wedge\tau}), t \geq 0$ is a Martingale.
5. Let $\lambda \in (0, \infty)$. The network $T_d(\lambda)$ is the graph T_d with the following weight function: $c(x, y) = \lambda^j$ if $d(0, y) = d(0, x) + 1 = j$ i.e., edge-weights are λ^j on edges between vertices at level $j - 1$ to j. Show that the Random Walk on $T_d(\lambda)$ is transient iff $d\lambda > 1$.

Notions and Notation-

T_d, **d-ary Tree:** For $d \geq 2$, let T_d be an infinite d-ary tree rooted at o i.e., o has d-neighbours, each of whom have d-further neighbours and so on. Level j vertices consist of those vertices at distance j from the root.

Conditional Variance: For random variable Y and a random element X, define $\mathsf{VAR}(Y \mid X) := \mathbb{E}\Big[\big(Y - \mathbb{E}[Y \mid X]\big)^2 \mid X\Big]$.

Sub-Gaussian random variable with variance factor ν: $\Psi_{X-\mu}(s) \leq s^2\nu/2$, for $s \in \mathbb{R}$ and where Ψ is the cumulant generating function i.e., $\Psi_X(s) = \log \mathbb{E}\big[e^{sX}\big]$.

Sub-exponential random variable with parameters ν and α: $\Psi_{X-\mu}(s) \leq s^2\nu/2$, for $|s| \leq \alpha^{-1}$.

Hölder's inequality: Let $w_j \geq 0$, $\sum_{j=1}^n w_j \leq 1$ be weights and Y_j's be non-negative random variables such that $\mathbb{E}\Big[Y_j^{1/w_j}\Big] < \infty$ for all j. Then, we have that

$$\mathbb{E}\left[\prod_{j=1}^n Y_j\right] \leq \prod_{j=1}^n \mathbb{E}\Big[Y_j^{1/w_j}\Big]^{w_j}.$$

Gamma function: $\Gamma(a) = \int_0^\infty e^{-s} s^{a-1}\, ds$ for $a > 0$ and $\Gamma(a) \leq a^a$.

Stirling's Approximation: $\sqrt{2\pi n}\, (\frac{n}{e})^n\, e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\, (\frac{n}{e})^n\, e^{1/(12n)}$.

Harmonic functions: For a bounded $f : V \to \mathbb{R}$ and stochastic matrix P, define the Laplacian operator as $\Delta f(x) := \sum_y P(x,y) f(y) - f(x)$. A function f is *harmonic on* $W \subset V$ if $\Delta f(x) = 0$ for all $x \in W$.

Erdős-Rényi random graph $G(n,p)$: This is the random sub-graph of the complete graph where every edge is retained with probability p and independently of each other.

Clique: Clique is a complete sub-graph. For example, a k-clique is a complete graph on k vertices and size of such a clique is k.